

Distances in inhomogeneous quintessence cosmology

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Abstract

We investigate the properties of cosmological distances in locally inhomogeneous universes with pressureless matter and dark energy (*quintessence*), with constant equation of state, $p_X = w_X \rho_X$, $-1 \leq w_X < 0$. We give exact solutions for angular diameter distances in the empty beam approximation. In this hypothesis, the distance-redshift equation is derived from the multiple lens-plane theory. The case of a flat universe is considered with particular attention. We show how this general scheme makes distances degenerate with respect to w_X and the smoothness parameters α_i , accounting for the homogeneously distributed fraction of energy of the i -components. We analyse how this degeneracy influences the critical redshift where the angular diameter distance takes its maximum, and put in evidence future prospects for measuring the smoothness parameter of the pressureless matter, α_M .

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1 Introduction

Recent measurements of the cosmic microwave background anisotropy (de Bernardis et al. 2000) strongly support the flat universe predicted by inflationary cosmology. At the same time, there are strong evidences, i.e. from dynamical estimates or X-ray and lensing observations of clusters of galaxies, that the today pressureless matter parameter of the universe Ω_M is significantly less than unity ($\Omega_M = \Omega_{CDM} + \Omega_B \simeq [0.30 \pm 0.10] + [0.04 \pm 0.01]$, where CDM and B respectively stand for cold dark matter and baryons), while the hot matter (photons and neutrinos) is really neglectable. Flatness so requires another positive contribute to the energy density of the universe, the so called dark energy – see, for example, Silveira & Waga (Silveira & Waga 1997) or Turner & White (Turner & White 1997) – that, according to the recent data from SNIa (Riess et al. 1998; Perlmutter et al. 1999), must have negative pressure to account for an accelerated expanding universe. One of the possible solutions of this puzzle is the Λ CDM universe where the fraction Ω_X of the dark energy is supported by a cosmological constant Λ . A very promising alternative, the quintessence, was proposed by Caldwell, Dave & Steinhardt (Caldwell et al. 1998) in the form of a dynamical, spatially inhomogeneous energy with negative pressure, i.e. the energy of a slowly evolving scalar field with positive potential energy. Apart from flatness, a scalar field coupled to matter through gravitation can also explain the so called coincidence problem, that is, it can provide a mechanism to make the today densities of matter and dark energy comparable. As shown in Wang et al. (Wang et al. 2000), a large set of independent observations agrees with a universe described in terms of pressureless matter and quintessence only, a hypothesis that we will make in this paper.

A scalar field is not an ideal adiabatic fluid (Grishchuk 1994; Caldwell et al. 1998) and the sound speed in it varies with the wavelength in such a way that high frequency modes remain stable still when $w_X < 0$. Moreover, smoothness is gauge dependent, and so a fluctuating inhomogeneous energy component is naturally defined (Caldwell et al. 1998). Inhomogeneities, both in quintessence and CDM, make the relations for the distances derived in Friedmann-Lemaître-Robertson-Walker (FLRW) models not immediately applicable to the interpretation of experimental data both in measurements of luminosity distances and angular diameter distances. The observed universe appears to be homogeneously distributed only on large scales ($\gtrsim 500$ Mpc), while the propagation of light is a local phenomena. In lack of a really satisfactory exact solution for inhomogeneous universes in the framework of General Relativity (Krasinski 1997), the usual, very simple framework we shall adopt for the study of distances is the *on average FLRW universe* (Schneider, Ehlers & Falco 1992; Seitz, Schneider & Ehlers 1994), where: *i*) the relations on a large scale are the same of the corresponding FLRW universe; *ii*) the anisotropic distortion of the bundle of light rays contributed by external inhomogeneities (the shear σ) is not significant; *iii*) only the fraction α_i , the so called smoothness parameter, of the *i*-component contributes to the Ricci focusing \mathcal{R} , that is to the isotropic focusing of the bundle. The distance recovered in this ‘empty beam approximation’, sometimes known as Dyer-Roeder (DR) distance, has been long studied (Zel’dovich 1964; Kantowski 1969; Dyer & Roeder 1972, 1973; Linder 1988) and

now is becoming established as a very useful tool for the interpretation of experimental data (Kantowski 1998; Kantowski & Thomas 2000; Perlmutter et al. 1999; Giovi, Occhionero & Amendola 2000). For a very large sample of standard candles, as SNIa for luminosity distance, or standard rods, as compact radio sources for angular diameter distances (Gurvits, Kellermann & Frey 1999), the effects of overdensity and underdensity along different lines of sight between observer and sources balance each other out, and the mean distance of the probability distribution of distances for fixed source redshift, corresponding to the FLRW distance, can be used (Wang 1999, 2000). Nevertheless, for the present quite poor samples, sources are very rarely brightened by the gravitational effect of occurring overdense clumps, and so, for propagation of light far from local inhomogeneities, the source appears dimmer and smaller with respect to the homogeneous case. So the distance to be used is the mode of the distribution, that is, in a very good approximation, the DR distance (Schneider et al. 1992; Kantowski 1998). In general, the smoothness parameters for matter and quintessence (respectively α_M and α_X) and the equation of state w_X are redshift dependent, but they can be considered, in the redshift interval covered by observations, as constant. For the clumpiness parameters, the constancy is motivated by the absence of significant variations in the development of structures in the observed redshift range. Instead, also in presence of redshift variations in w_X , only its integral properties have effect on the distance. In fact, in flat FLRW models the distance depends on w_X only through a triple integral on the redshift (Maor, Brustein & Steinhardt 2000). Hereafter, we will consider the α_i and w_X as constant. For the cosmological constant $w_X = -1$; a network of light, nonintercommuting topological defects (Vilenkin 1984; Spergel & Pen 1997) gives $w_X = -m/3$ where m is the dimension of the defect: for a string $w_X = -1/3$, for a domain wall $w_X = -2/3$; an accelerated universe requires $w_X < -1/3$.

In this approximation, distances are functions of a family of parameters: Ω_M and Ω_X , which describe the energy content of the universe on large scales; w_X , which describes the equation of state of the quintessence and varies between -1 and 0 ; two clumpiness parameters, α_M and α_X , representing the degree of homogeneity of the universe and which are to be used for phenomena of local propagation.

This paper is organized as follows: in Sect.2 we introduce the so called DR equation and discuss some features and its solution for quintessence in the form of topological defects; Sect.3 lists the general solution for the DR equation in the flat case and some simple expressions in extreme situations; in Sect.4 the multiple lens-plane theory of gravitational lensing is applied for a derivation of the DR equation without using the focusing equation; Sect.5 is devoted to the degeneracy of the distance with respect to the various parameters; in Sect.6 the critical redshift at which the angular diameter distance takes its maximum is studied, and, at last, in Sect.7 we draw the conclusions.

2 The beam equation for inhomogeneous quintessence

In the hypotheses discussed above, the focusing equation (Sachs 1961; Schneider et al. 1992) for the angular diameter distance D_A in terms of an affine parameter λ ,

$$\frac{d^2 D_A}{d\lambda^2} = -(|\sigma(\lambda)|^2 - \mathcal{R}(\lambda))D_A, \quad (1)$$

becomes (see also Linder 1988)

$$\frac{d^2 D_A}{d\lambda^2} + \frac{1}{2}(1+z)^2 \left[3\alpha_M \Omega_M (1+z)^3 + n_X \alpha_X \Omega_X (1+z)^{n_X} \right] D_A = 0, \quad (2)$$

where $n_X \equiv 3(w_X + 1)$ ($0 < n_X < 3$), and the relation between λ and the redshift z , in terms of the generalized Hubble parameter

$$H(z) = H_0 \sqrt{\Omega_M (1+z)^3 + \Omega_X (1+z)^{n_X} + \Omega_K (1+z)^2}, \quad \Omega_K \equiv 1 - \Omega_M - \Omega_X, \quad (3)$$

is

$$\frac{dz}{d\lambda} = (1+z)^2 \frac{H(z)}{H_0}. \quad (4)$$

The isotropic focusing effect in equation (2) is simply represented by the multiplicative factor to D_A ; this coefficient increases with α_M , α_X and n_X .

Equation (2) is sometimes called the generalized DR equation. Substituting for λ in equation (2) by using equation (4), we have

$$H^2(z) \frac{d^2 D_A}{dz^2} + \left[\frac{2H^2(z)}{1+z} + \frac{1}{2} \frac{dH^2}{dz} \right] \frac{D_A}{dz} + \frac{1}{2}(1+z) \left[3\alpha_M \Omega_M + n_X \alpha_X \Omega_X (1+z)^{n_X-3} \right] D_A = 0; \quad (5)$$

the initial conditions on equation (5) are

$$D_A(z_d, z_d) = 0, \quad (6)$$

$$\left. \frac{d}{dz} D_A(z_d, z) \right|_{z=z_d} = \frac{1}{1+z_d} \frac{c}{H(z_d)},$$

where $D_A(z_d, z)$ is the angular diameter distance between z_d (that, in general, can be different from zero, as occurs in gravitational lensing for the deflector) and the source at z .

Changing to the expansion factor $a \equiv 1/(1+z)$, equation (5) is

$$\begin{aligned} a^2 \left[\Omega_M + \Omega_X a^{3-n_X} + \Omega_K a \right] \frac{d^2 D_A}{da^2} - a \left[\frac{3}{2} \Omega_M + \frac{n_X}{2} \Omega_X a^{3-n_X} + \Omega_K a \right] \frac{dD_A}{da} \\ + \left[\alpha_M \frac{3}{2} \Omega_M + \alpha_X \frac{n_X}{2} \Omega_X a^{3-n_X} \right] D_A = 0, \end{aligned} \quad (7)$$

a form which will be useful in the next sections.

In a generic space-time, the angular diameter distance D_A and the luminosity distance D_L are related by (Etherington 1933)

$$D_L = (1+z)^2 D_A, \quad (8)$$

so that the considerations we will make about D_A are easily extended to D_L .

2.1 Exact solutions for $\Omega_K \neq 0$

The observational data nowadays available are in agreement with the hypothesis of a flat universe, but are also compatible with a non zero, although small, value of Ω_K . A small value of Ω_K is also allowed by the inflationary theory. These circumstances make useful the study of the effect of the curvature on the cosmological distances since today technology allows to put strong constraints on the cosmological parameters.

For $\alpha_X = 1$, equation (7) reduces to

$$a^2(\Omega_M + \Omega_K a + \Omega_X a^{3-n_X}) \frac{d^2 D_A}{da^2} - a \left(\frac{3}{2} \Omega_M + \Omega_K a + \frac{n_X}{2} \Omega_X a^{3-n_X} \right) \frac{dD_A}{da} + \left(\frac{3}{2} \alpha_M + \frac{n_X}{2} \Omega_X a^{3-n_X} \right) D_A = 0. \quad (9)$$

To solve equation (9), we proceed as in Demianski et al. (Demianski et al. 2000). First, we look for a solution in the power form a^s when $\Omega_X = \Omega_K = 0$. The parameter s is constrained to fulfill the algebraic equation

$$s^2 - \frac{5}{2}s + \frac{3}{2}\alpha_M = 0, \quad (10)$$

which has the solutions

$$s_{\pm} = \frac{5}{4} \pm \frac{1}{4} \sqrt{25 - 24\alpha_M} \equiv \frac{5}{4} \pm \beta. \quad (11)$$

When $\Omega_X \neq 0$, $\Omega_K \neq 0$, we choose to impose the form $D_A = a^s f(a)$ to the solution, being f a generic function. Inserting this expression into equation (9) we have for f

$$a \left(\Omega_M + \Omega_K a + \Omega_X a^{3-n_X} \right) \frac{d^2 f}{da^2} - \frac{df}{da} \left(\Omega_M (2s - \frac{3}{2}) + \Omega_K (2s - 1)a + \left(s(s - 1) - \frac{n_X}{2}s \right) \Omega_X a^{3-n_X} \right) + (\Omega_K s(s - 2) + \Omega_X (s^2 - s(1 + \frac{n_X}{2}))) a^{2-n_X} f = 0. \quad (12)$$

The initial conditions at $a = 1$ for the auxiliary function f come out from equation (6) evaluated at $z = 0$,

$$f(1) = 0, \quad (13)$$

$$\left. \frac{d}{dz} f(a) \right|_{a=1} = \frac{c}{H_0}.$$

Equation (13) is very useful to obtain some exact solutions of the DR equation, corresponding to integer values of the quintessence parameter n_X . Since the solutions for the case of the cosmological constant, $n_X = 0$ (Kantowski 1998), and for the pressureless matter, $n_X = 3$ (Seitz & Schneider 1994), are already known, we will consider only string networks, $n_X = 2$ ($w_X = -1/3$) and the domain walls with $n_X = 1$ ($w_X = -2/3$). Let us start with the case $n_X = 2$, when equation (13) reduces to

$$a(c_1 + c_2 a) \frac{d^2 f}{da^2} + (c_3 + c_2 c_4 a) \frac{df}{da} + c_5 f = 0, \quad (14)$$

being

$$\begin{aligned}
c_1 &= \Omega_M, \\
c_2 &= \Omega_K + \Omega_X, \\
c_3 &= c_2 \left(2s - \frac{3}{2} \right), \\
c_4 &= 2s - 1, \\
c_5 &= \Omega_K s(s-2) + \Omega_X \left(s \left(s - \frac{3}{2} + \frac{1}{2} \right) \right).
\end{aligned} \tag{15}$$

Equation (14) is of hypergeometric type, i.e. it has three regular singularities (Ince 1956), and so, for $n_X = 2$, f is the hypergeometric function. If we indicate with f_{s_+} and f_{s_-} two independent solutions for, respectively, $s = s_+$ and $s = s_-$, we can write the general solution of equation (9) for $n_X = 2$ as

$$\begin{aligned}
D_A &= A_+ a^{s_+} f_{s_+}(a) + A_- a^{s_-} f_{s_-}(a) \\
&= \frac{1}{(1+z)^{5/4}} \left(A_+ (1+z)^{-\beta} f_{s_+} \left(\frac{1}{1+z} \right) + A_- (1+z)^{\beta} f_{s_-} \left(\frac{1}{1+z} \right) \right),
\end{aligned} \tag{16}$$

where A_+ and A_- are constants determined by the initial conditions. In equation (16) we have expressed the scale factor, a , in terms of the redshift.

Let us consider now the case $n_X = 1$, when the equation for f becomes

$$a \left(\frac{\Omega_M}{\Omega_X} + \frac{\Omega_K}{\Omega_X} a + a^2 \right) \frac{d^2 f}{da^2} \tag{17}$$

$$+ \left(\frac{\Omega_M}{\Omega_X} \left(2s - \frac{3}{2} \right) + \frac{\Omega_K}{\Omega_X} (2s-1) a + \left(2s - \frac{1}{2} \right) a^2 \right) \frac{df}{da} \tag{18}$$

$$+ \left(\frac{\Omega_K}{\Omega_X} s(s-2) + \left(s(s-2) + \frac{1}{2} \right) a \right) f = 0. \tag{19}$$

Equation (17) is a fuchsian equation with three finite regular points plus a regular singularity at ∞ (Ince 1956). The regular points in the finite part of the complex plane are

$$\begin{aligned}
a_1 &= 0, \\
a_2 &= \frac{-\Omega_K - \sqrt{\Omega_K^2 - \Omega_M \Omega_X}}{2}, \\
a_3 &= \frac{-\Omega_K + \sqrt{\Omega_K^2 - \Omega_M \Omega_X}}{2}.
\end{aligned} \tag{20}$$

The trasformation $y = a/a_2$ sends $a_2 \rightarrow 1$ and $a_2 \rightarrow \zeta = \frac{a_3}{a_2}$. In terms of y , equation (17) is

$$\begin{aligned}
&y(y-1)(y-\zeta) \frac{d^2 f dy^2}{+} \left(\frac{\Omega_M}{\Omega_X} (2s - \frac{3}{2}) + \frac{\Omega_K}{\Omega_X} (2s-1) a_2 y \right) \\
&+ \left((2s - \frac{1}{2}) a_2^2 y^2 \right) \frac{df}{dy} + \left(\frac{\Omega_K}{a_2 \Omega_X} s(s-2) + \left(s(s-2) + \frac{1}{2} \right) a_2 y \right) f = 0,
\end{aligned} \tag{21}$$

which can be reduced to the standard form

$$\frac{d^2 f}{dy^2} + \left(\frac{\gamma}{y} + \frac{\delta}{y-1} + \frac{\epsilon}{y-\zeta} \right) \frac{df}{dy} + \left(\frac{\theta\lambda y - q}{y(y-1)(y-\zeta)} \right) f = 0, \quad (22)$$

where

$$\begin{aligned} \gamma + \delta &= \left(2s - \frac{1}{2} \right) a_2^2, \\ \frac{\Omega_K}{\Omega_X} (2s-1) a_2 &= \gamma(1+\zeta) + \delta\zeta + \epsilon, \\ \gamma\zeta &= \frac{\Omega_M}{\Omega_X} \left(2s - \frac{3}{2} \right), \\ q &= \frac{\Omega_K}{\Omega_M} s(s-2), \\ \theta\lambda &= s \left(s - \frac{3}{2} \right) + \frac{1}{2}. \end{aligned} \quad (23)$$

Equation (22) is the Heun equation (Erdélyi et al. 1955), which is slightly more complicated than the hypergeometric equation, possessing four points of regular singularity in the entire complex plain, rather than three. The constant q is the so called *accessory parameter*, whose presence is due to the fact that a fuchsian equation is not completely determined by the position of the singularities and the indices. The Heun equation can be characterized by a \mathcal{P} symbol, and the solutions can be expanded in series of hypergeometric functions. Thus, the solution of the equation (9) for $n_X = 1$ can be formally written as equation (16), once the functions f_{s+} and f_{s-} are interpreted as Heun functions.

In Fig. ??, we plot the solutions found above.

3 Exact solutions for $\Omega_K = 0$

The DR equation for a flat universe has already been solved considering the limiting case of the cosmological constant (Kantowski 1998; Kantowski & Thomas 2000; Demianski et al. 2000). Here, in presence of generic quintessence, we propose the general solution in terms of hypergeometric functions and, then, list particular solutions in terms of elementary functions.

3.1 General solution

When $\Omega_K = 0$, equation (7) reduces to

$$\begin{aligned} &\left(a^2(\Omega_M + (1 - \Omega_M)a^{3-n_X}) \right) \frac{d^2 D_A}{da^2} - a \left(\frac{3}{2}\Omega_M + \frac{n_X}{2}(1 - \Omega_M)a^{3-n_X} \right) \frac{dD_A}{da} \\ &+ \left(\frac{3}{2}\alpha_M\Omega_M + \frac{n_X}{2}\alpha_X(1 - \Omega_M)a^{3-n_X} \right) D_A = 0; \end{aligned} \quad (24)$$

dividing equation (24) by Ω_M and defining $\mu \equiv \frac{1-\Omega_M}{\Omega_M}$, we have

$$a^2(1+\mu a^{3-n_X})\frac{d^2 D_A}{da^2} - a\left(\frac{3}{2} + \frac{n_X}{2}\mu a^{3-n_X}\right)\frac{dD_A}{da} + \left(\frac{3}{2}\alpha_M + \frac{n_X}{2}\alpha_X\mu a^{3-n_X}\right)D_A = 0. \quad (25)$$

To solve equation (25), we proceed as in Sect.2.1. First, we look for a solution in the power form a^s when $\mu = 0$. The parameter s is constrained to fulfill equation (10). When $\mu \neq 0$, we choose to impose the form $D_A = a^s f(a)$ to the solution, where f is generic. Inserting this expression into equation (25) and changing to $x \equiv a^{3-n_X}$, we have for f

$$\begin{aligned} & x(3-n_X)(1+\mu x)\frac{d^2 f}{dx^2} + \left(\left(2s-n_X+\frac{1}{2}\right) + \left(2-\frac{3}{2}n_X+2s\right)\mu x\right)\frac{df}{dx} \\ & + \frac{\mu}{2}\left(s - \frac{3\alpha_M - n_X\alpha_X}{3-n_X}\right)f = 0. \end{aligned} \quad (26)$$

Again, this equation can be solved in terms of hypergeometric functions. Denoting with f_{s+} and f_{s-} two of such independent solutions for, respectively, $s = s_+$ and $s = s_-$, we can write the general solution of equation (24) as

$$\begin{aligned} D_A &= A_+ a^{s_+} f_{s_+}(x(a)) + A_- a^{s_-} f_{s_-}(x(a)) \\ &= \frac{1}{(1+z)^{5/4}} \left(A_+ (1+z)^{-\beta} f_{s_+} \left(\frac{1}{(1+z)^{3-n_X}} \right) + A_- (1+z)^{\beta} f_{s_-} \left(\frac{1}{(1+z)^{3-n_X}} \right) \right), \end{aligned} \quad (27)$$

where A_+ and A_- are constants determined by the initial conditions.

3.2 Particular cases

Once we have the general solution of equation (24) in terms of hypergeometric functions, we go now to list some expressions of the angular diameter distance in terms of elementary functions in two extremal cases.

3.2.1 Homogeneous universe

In this case we have that $\alpha_M = \alpha_X = 1$, so that the angular diameter distance takes the form valid in a FLRW universe, that is

$$D_A(z_d, z) = \frac{c}{H_0} \frac{1}{1+z} \int_{z_d}^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + (1-\Omega_M)(1+z')^{n_X}}}. \quad (28)$$

This is the integral of the differential binomial

$$x^\mu (a + bx^\nu)^\rho, \quad (29)$$

where $x = 1 + z$, $a = \Omega_M$, $b = 1 - \Omega_M$, $\mu = -3/2$, $\nu = n_X - 3$ and $\rho = -1/2$. We can put this integral in rational form when

$$n_X = \frac{3s-1}{s}, \quad s \in \mathcal{Z} - \{0\}, \quad (30)$$

performing the substitutions (Picone & Miranda 1943)

$$t = \sqrt{\Omega_M + (1 + \Omega_M)(1 + z)^{n_X-3}}$$

when s is even and

$$t = \sqrt{\frac{\Omega_M + (1 + \Omega_M)(1 + z)^{n_X-3}}{(1 + z)^{n_X-3}}}$$

for odd s . Equation (30) includes all and only the rational values of n_X for which equation (28) can be solved in terms of elementary functions. n_X varies from 2 ($w_X = -1/3$), when quintessence evolves like curvature, to 4 ($w_X = 1/3$) (*hot dark matter*); for $s \rightarrow \pm\infty$, n_X tends to 3, giving ordinary pressureless matter. For $n_X = 2$ (see also Lima & Alcaniz 2000) we get

$$D_A(z_d, z) = \frac{c}{H_0} \frac{2}{(1 + z)\sqrt{1 - \Omega_M}} \left[\text{Arctanh} \left(\frac{\sqrt{1 + \Omega_M z}}{\sqrt{1 - \Omega_M}} \right) \right]_{z=z_d}^z; \quad (31)$$

we note that with respect to the dynamical equations, a flat universe with $n_X = 2$ behaves like an open one with $\Omega_K = 1 - \Omega_M \neq 0$, but, on the other hand, while quintessence contributes to the Ricci focusing, a geometric term does not. For $n_X = 4$, it is

$$D_A(z_d, z) = \frac{c}{H_0} \frac{2}{\Omega_M(1 + z)} \left[\frac{\sqrt{1 + z - \Omega_M z}}{(1 + z)^{1/2}} \right]_z^{z=z_d}. \quad (32)$$

Equation (32) holds in the past history of the universe at the epoch of matter-radiation equality ($z_{eq} \sim 10^4$). Other solutions with $2 < n_X < 4$ are easily found. Even if they can be physically interesting when related to other behaviours of the scale factor, they cannot explain the today observed accelerated universe. So, we will not mention them here.

3.2.2 Totally clumpy universes

We now study very particular models of universe in which both matter and quintessence are totally clumped, that is $\alpha_M = \alpha_X = 0$. In this case, the DR equation reduces to a first order equation and the expression for the angular diameter distance becomes

$$D_A(z_d, z) = \frac{c}{H_0} (1 + z_d) \int_{z_d}^z \frac{dz'}{(1 + z')^2 \sqrt{\Omega_M(1 + z')^3 + \Omega_X(1 + z')^{n_X}}}. \quad (33)$$

When $n_X = 0$ ($w_X = -1$) and $\alpha_M = 0$, the DR equation becomes of the first order independently of the values of Ω_X and α_X , and so the distance takes the form

$$D_A(z_d, z) = \frac{c}{H_0} (1 + z_d) \int_{z_d}^z \frac{dz'}{(1 + z')^2 \sqrt{\Omega_M(1 + z')^3 + \Omega_X}}. \quad (34)$$

Once again, in equation (33) there is the integral of a differential binomial of the form given in equation (29), with, this time, $a = \Omega_M$, $b = 1 - \Omega_M$, $\mu = -7/2$, $\nu = n_X - 3$ and $\rho = -1/2$. When n_X is rational, all and only the solutions of equation (33) in terms of elementary functions occur when

$$n_X = \frac{3s-5}{s}, \quad s \in \mathcal{Z} - \{0\}; \quad (35)$$

for any such s we can perform the same substitutions already described for homogeneous universes in the previous subsection. Now, we have values of $n_X < 2$: for $s = 2, 3, 4$, respectively, we find n_X (w_X) = $1/2$ ($-5/6$), $4/3$ ($-5/9$), $7/4$ ($-5/12$). For $n_X = 1/2$, it is

$$D_A(z_d, z) = \frac{c}{H_0} \frac{4(1+z_d)}{5(1-\Omega_M)} \left(\sqrt{\Omega_M + \frac{1-\Omega_M}{(1+z_d)^{5/2}}} - \sqrt{\Omega_M + \frac{1-\Omega_M}{(1+z)^{5/2}}} \right); \quad (36)$$

for $n_X = 4/3$, we get

$$\begin{aligned} D_A(z_d, z) &= \frac{c}{H_0} (1+z_d) \left[\frac{6}{5} \Omega_M \sqrt{1-\Omega_M + \Omega_M(1+z)^{5/3}} \left((\Omega_M - 1)^2 + \right. \right. \\ &\quad \left. \left. + \frac{2}{3} (\Omega_M - 1)^2 \left(1 - \Omega_M + \Omega_M(1+z)^{5/3} \right) + \frac{1}{5} \left(1 - \Omega_M + \Omega_M(1+z)^{5/3} \right)^2 \right) \right]_{z=z_d}^z, \end{aligned} \quad (37)$$

and, for $n_X = 7/4$,

$$D_A(z) = \frac{c}{H_0} \frac{8}{5(1-\Omega_M)^2} \left(\left(\frac{1}{3} - \Omega_M \right) - \left(\frac{1}{3} \left(\frac{1-\Omega_M}{(1+z)^{5/4}} + \Omega_M \right)^{3/2} \right. \right. \quad (38)$$

$$\left. - \Omega_M \sqrt{\frac{1-\Omega_M}{(1+z)^{5/4}} + \Omega_M} \right). \quad (39)$$

Other interesting results are obtained when $n_X = 2$ ($s = 5$) and $n_X = 4$ ($s = -5$). For $n_X = 2$ (string networks), the angular diameter distance is

$$\begin{aligned} D_A(z_d, z) &= 2 \frac{c}{H_0} (1+z_d) \Omega_M^2 \mathcal{E} \left[(\Omega_M - 1)^3 + (\Omega_M - 1)^2 \mathcal{E} \right. \\ &\quad \left. + \frac{3}{5} (\Omega_M - 1) \mathcal{E}^2 + \frac{1}{7} \mathcal{E}^3 \right]_{z=z_d}^z, \end{aligned} \quad (40)$$

where

$$\mathcal{E} = \sqrt{1 - \Omega_M + (1+z)\Omega_M};$$

for $n_X = 4$ (hot dark matter), it is

$$D_A(z_d, z) = \frac{c}{H_0} \frac{1+z_d}{\Omega_M^3 (\Omega_M - 1)} \left[\frac{\text{Arctan} \left(\sqrt{\frac{1+z-\Omega_M z}{(1+z)(\Omega_M - 1)}} \right)}{\sqrt{1-\Omega_M}} + \sqrt{\frac{(1+z)(1+z-z\Omega_M)}{\Omega_M}} \right]_z^{z=z_d}. \quad (41)$$

In the limit $s \rightarrow \pm\infty$, n_X goes to 3 (CDM).

4 An alternative derivation of the generalized DR equation

As already shown for a universe with pressureless matter (Schneider & Weiss 1988; Schneider et al. 1992), it is possible to derive the DR equation from the multiple lens-plane theory, without referring to the focusing equation. We want, in the framework of the on average FLRW universes, to generalize this result to the case of inhomogeneous quintessence. The basic idea is the simulation of the clumpiness by adding to a smooth homogeneous background a hypothetical density distribution of zero total mass, which is made of two components: a distribution of clumps (both in dust and dark energy) and a uniform negative energy density such that the mean density of the sum of both components is zero. After such addition, the average properties of the universe on large scales are still that corresponding to the background FLRW model. The gravitational surface density Σ of clumps in a shell of size Δz centered on the observer is then

$$\Sigma = \Delta z \frac{dr_{prop}}{dz} T_{cl}^{00}, \quad (42)$$

where the relation between the redshift and the proper distance r_{prop} is that valid in FLRW universes,

$$\frac{dr_{prop}}{dz} = \frac{c}{H(z)} \frac{1}{1+z}, \quad (43)$$

and T_{cl}^{00} is the 0-0 element of the total energy-momentum tensor in clumps,

$$T_{cl}^{00}(z) = (1 - \alpha_M) T_M^{00} + (1 - \alpha_X) T_X^{00} = (1 - \alpha_M) \rho_M(z) c^2 + (1 - \alpha_X) (1 + w_X) \rho_X(z) c^2. \quad (44)$$

In an on average FLRW universe, the densities of pressureless matter and quintessence are, respectively,

$$\begin{aligned} \rho_M(z) &= (1+z)^3 \Omega_M \rho_{cr}, \\ \rho_X(z) &= (1+z)^{n_X} \Omega_X \rho_{cr}, \end{aligned} \quad (45)$$

where $\rho_{cr} \equiv 3H_0^2/(8\pi G)$ is the today critical density. The dimensionless surface density k corresponding to equation (42) is

$$\begin{aligned} k\Delta z \equiv \frac{4\pi G}{c^2} \frac{D_1(z) D_1(z, z_s)}{D_1(z_s)} \Sigma &= \frac{H_0^2}{cH(z)} \frac{(1+z)^3}{2} [3(1 - \alpha_M) \Omega_M + \\ &+ n_X (1 - \alpha_X) \Omega_X (1+z)^{n_X-3}] \frac{D_1(z) D_1(z, z_s)}{D_1(z_s)} \Delta z, \end{aligned} \quad (46)$$

where the subscript 1 refers to diameter angular distances in FLRW universes and z_s is a hypothetical source redshift. The so constructed spherical shells will act as multiple lens-plane. The ray-trace equation which describes successive deflections caused by a series of lens planes is (Schneider et al. 1992)

$$\mathbf{x}_j = \mathbf{x}_1 - \sum_{i=1}^{j-1} \frac{D_1(z_i, z_j)}{D_1(z_j)} \hat{\alpha}_i, \quad (47)$$

where \mathbf{x}_i is the bidimensional angular position vector in the i th lens plane and $\hat{\boldsymbol{\alpha}}_i$ is the deflection angle a light ray undergoes if it traverses the i th lens plane at \mathbf{x}_i . The lens planes are ordered such that $z_i < z_j$ if $i < j$. The solid angle distortion is described by the 2×2 Jacobian matrices of the mapping equation (47),

$$A_i \equiv \frac{\partial \mathbf{x}_i}{\partial \mathbf{x}_1}, \quad (48)$$

and by the derivatives of the scaled deflection angle $\boldsymbol{\alpha}_i = (D_1(z_i, z_s)/D_1(z_s))\hat{\boldsymbol{\alpha}}_i$,

$$U_i \equiv \frac{\partial \boldsymbol{\alpha}_i}{\partial \mathbf{x}_i}. \quad (49)$$

By taking the derivative of equation (47) with respect to the independent variable \mathbf{x}_1 , which represents the angular position of an image on the observer sky, we have the recursion relation

$$A_j = \mathcal{I} - \sum_{i=1}^{j-1} \frac{D_1(z_i, z_j)D_1(z_s)}{D_1(z_j)D_1(z_i, z_s)} U_i A_i, \quad (50)$$

with $A_1 = \mathcal{I}$, \mathcal{I} being the two-dimensional identity matrix. In our model of a clumpy universe, the matrices U_i are given by (Schneider & Weiss 1988)

$$U_i = -k_i \mathcal{I} \Delta z - \mathcal{T}_i, \quad (51)$$

where the first term accounts for the negative convergence caused by the smooth negative surface density and \mathcal{T}_i is the matrix that describes the deflection caused by the clumps. In the empty beam approximation (light rays propagating far away from clumps and vanishing shear), it is $\mathcal{T}_i = 0$ and then all the A_i are diagonal, $A_i = a_i \mathcal{I}$. Equation (50) becomes

$$a_j = 1 + \sum_{i=1}^{j-1} \frac{D_1(z_i, z_j)D_1(z_s)}{D_1(z_j)D_1(z_i, z_s)} k_i a_i \Delta z, \quad (52)$$

where the dependence on z_s drops out in the product of the ratio of distances by k_i . In the continuum limit, $\Delta z \rightarrow 0$, equation (52) is

$$a(z) = 1 + \int_0^z \frac{D_1(y, z)D_1(z_s)}{D_1(z)D_1(y, z_s)} k(y) a(y) dy. \quad (53)$$

Multiplying equation (52) by $D_1(z)$ and letting $D_A(z) = a(z)D_1(z)$, we obtain, substituting for the explicit expression of k given in equation (46),

$$D(z) = D_1(z) + \left(\frac{H_0^2}{c} \right) \int_0^z \frac{(1+y)^2}{H(y)} \left[\frac{3}{2}(1 - \alpha_M)\Omega_M + \frac{n_X}{2}(1 - \alpha_X)\Omega_X(1+y)^{n_X-3} \right] D_1(y, z) D_A(z) dy.$$

It is easy to verify that equation (54) is equivalent to the generalized DR equation equation (5) with initial condition given for $z_d = 0$. Changing to z_d the lower limit of the integration in equation (54) and $D_A(z)$ ($D_1(z)$) with $D_A(z_d, z)$ ($D_1(z_d, z)$), we have the equation for generic initial conditions. Equation (54), already derived with a different way of proceeding by Linder (Linder 1988), has here been found only using the multiple lens-plane theory.

Equation (54) is a Volterra integral equation of the second kind (Tricomi 1985) whose solution is

$$D_A(z) = D_1(z) + \int_0^z H(y, z) D_1(y) dy, \quad (54)$$

where the resolvent kernel $H(y, z)$ is given by the series of iterated kernels

$$H(y, z) = \sum_{i=0}^{\infty} K_i(y, z), \quad (55)$$

with

$$K_1(y, z) = \begin{cases} \left(\frac{H_0^2}{c} \right) \frac{(1+y)^2}{H(y)} \left[\frac{3}{2}(1 - \alpha_M) \Omega_M + \frac{n_X}{2}(1 - \alpha_X) \Omega_X (1+y)^{n_X-3} \right] D_1(y, z) & \text{if } y \leq z, \\ 0 & \text{elsewhere,} \end{cases}$$

and the iterated kernels K_i defined by the recurrence formula

$$K_{i+1}(y, z) \equiv \int_0^y K(y, x) K_i(x, z) dx. \quad (56)$$

Since for all i , $K_i(y, z)$ and $D_1(y)$ are no negative, we see from equations (54)-(56) that the diameter angular distance $D_A(z)$ is a decreasing function of both α_M and α_X ,

$$D_A(z, \alpha_M^{(1)}) \leq D_A(z, \alpha_M^{(2)}) \text{ for } \alpha_M^{(1)} \geq \alpha_M^{(2)}, \quad (57)$$

$$D_A(z, \alpha_X^{(1)}) \leq D_A(z, \alpha_X^{(2)}) \text{ for } \alpha_X^{(1)} \geq \alpha_X^{(2)}.$$

5 Parameter degeneration

As seen, the consideration of the DR equation in its full generality, with respect to the case of a homogeneous cosmological constant, demands the introduction of new parameters. Let us study the case of homogeneous dark energy ($\alpha_X = 1$). For $\alpha_X = 1$, equation (54), in units of c/H_0 , simplifies to

$$D_A(z) = D_1(z) + \int_0^z \sum_{i=1}^{\infty} K_i(y, z) D_1(y) dy, \quad (58)$$

while equation (56) reduces, for $y \leq z$, to

$$K_1(y, z) = \frac{3}{2}(1 - \alpha_M) \Omega_M \frac{H_0}{H(y)} (1+y)^2 D_1(y, z). \quad (59)$$

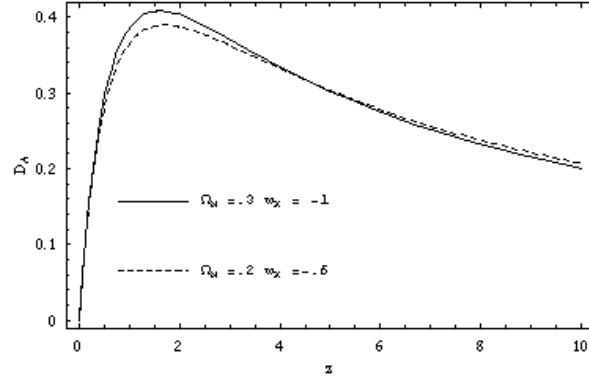


Figure 1: The angular diameter distance for two different FLRW universes. The unit of distance is taken to be c/H_0 .

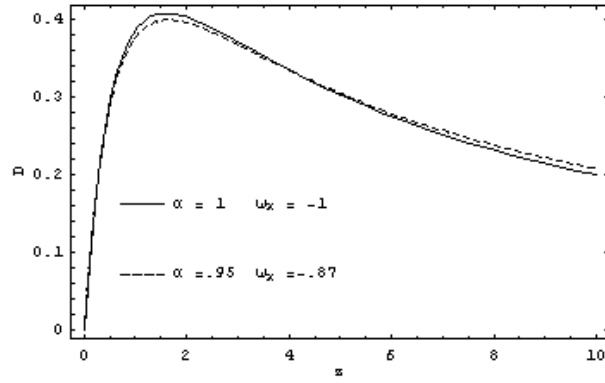


Figure 2: The angular diameter distance for different values of α_M and w_X (Ω_M is equal to 0.3). The unit of distance is taken to be c/H_0 .

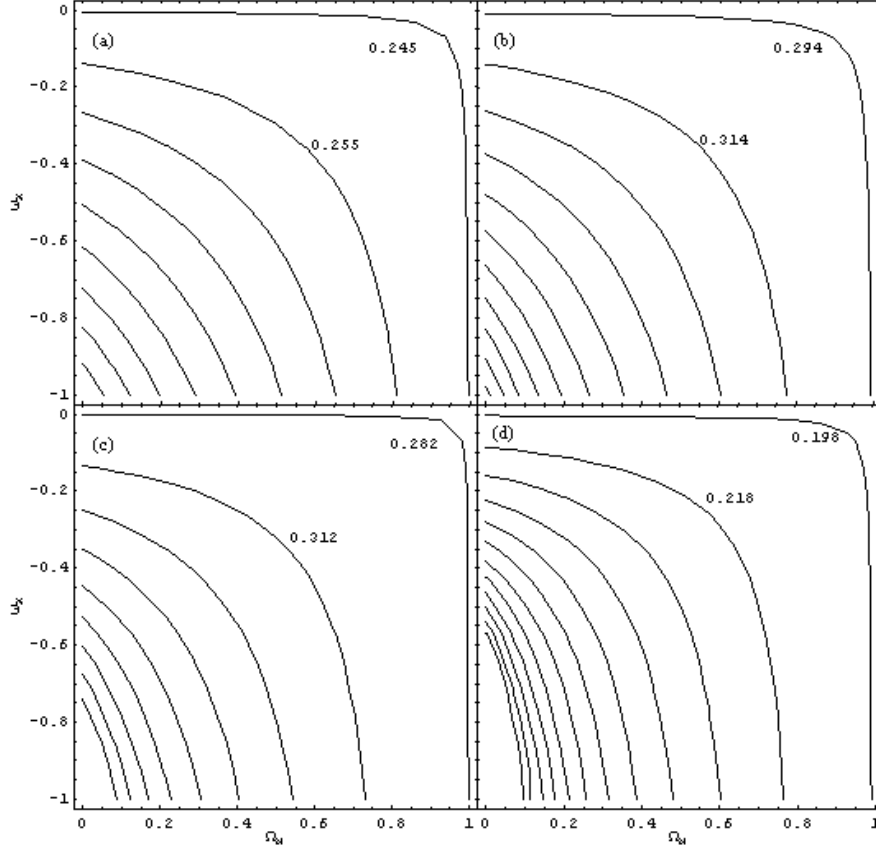


Figure 3: The angular diameter distance in the $\Omega_M - w_X$ plane, when $\alpha_M = \alpha_X = 1$. The distance increases from the top-right to the bottom-left corner. *a)* We assume $z = 0.5$; each contour is drawn with steps of 0.01. *b)* We assume $z = 1$; the step is 0.02. *c)* We assume $z = 2$; the step is 0.03. *d)* We assume $z = 5$; the step is 0.02. The unit of distance is taken to be c/H_0 .

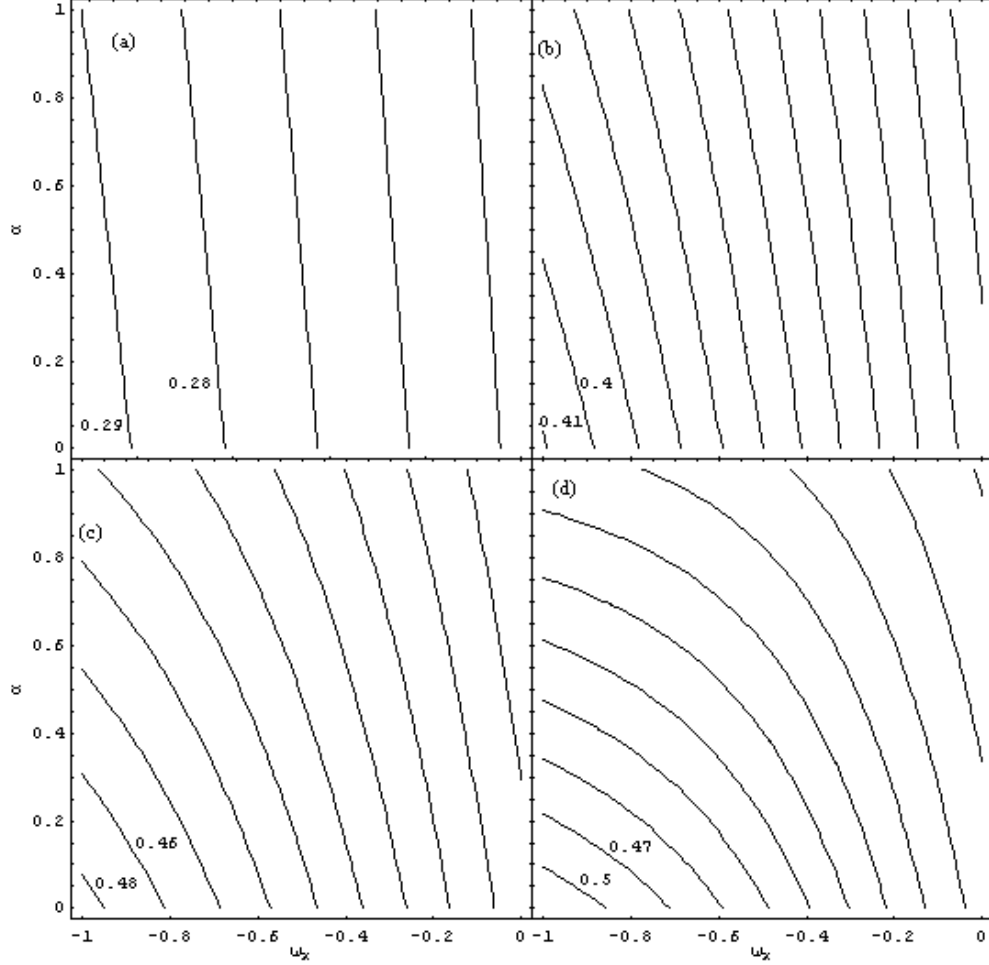


Figure 4: The angular diameter distance in the $w_X - \alpha_M$ plane, when $\Omega_M = 1$ and $\alpha_X = 1$. The distance increases from the top-right to the bottom-left corner. *a)* We assume $z = 0.5$; each contour is drawn with steps of 0.01. *b)* We assume $z = 1$; the step is 0.01. *c)* It is $z = 2$; the step is 0.02. *d)* it is $z = 5$; the step is 0.03. The unit of distance is taken to be c/H_0 .

Some monotonical properties with respect to the cosmological parameters are then easily derived. For an accelerated universe ($w_X < -1/3$), it is

$$\frac{\partial}{\partial \Omega_X} \frac{1}{H(z)} > 0, \quad \frac{\partial}{\partial \Omega_X} D_1(z) > 0 \text{ if } w_X < -1/3, \quad (60)$$

and so, for every value of the clumpiness parameter α_M , the angular diameter distance increases with increasing Ω_X ,

$$\frac{\partial}{\partial \Omega_X} D_A(z) > 0 \text{ if } w_X < -1/3. \quad (61)$$

When $w_X > -1/3$, the inequalities in equation (60) are reversed and the distance decreases with increasing Ω_X . With respect to the equation of state w_X , it is

$$\frac{\partial}{\partial w_X} \frac{1}{H(z)} < 0, \quad \frac{\partial}{\partial w_X} D_1(z) < 0; \quad (62)$$

and so

$$\frac{\partial}{\partial w_X} D_A(z) < 0; \quad (63)$$

large values of the distances correspond to large negative values of the pressure of quintessence so that, for fixed Ω_M , Ω_X and α_M , the angular diameter distance takes its maximum when the dark energy is in the form of a cosmological constant.

We now want to stress the dependences of the angular diameter distance on w_X , Ω_M and α_M in flat universes with $\alpha_X = 1$. As Fig.1 and Fig.2 show, the angular diameter distance is degenerate with respect to different pairs of parameters, since the distance in the Λ CDM model with $\Omega_M = 0.3$ is not distinguishable, within the current experimental accuracy (Perlmutter et al. 1999), from the one in a FLRW universe with less pressureless matter but a greater value of w_X or from an inhomogeneous universe with greater w_X and the same content of matter.

In Fig.3, we plot the degenerate values of the distance in the $\Omega_M - w_X$ plane when universe is homogeneous for four different source redshifts: as expected, the dependence of the distance on the cosmological parameters increases with the redshift of the source. A general feature is that the distance is less sensitive to the components of the universe when Ω_M is near unity and w_X goes to 0. This is easily explained: when Ω_M is large, quintessence density Ω_X is not, and the pressureless matter characterizes almost completely the universe; moreover, a value of w_X near zero describes a dark energy with an equation of state very similar to that of the ordinary matter. So, increasing w_X mimics a growth in Ω_M . On the other side, for low values of Ω_M (w_X) the distance is very sensitive to w_X (Ω_M) and this effect increases with the redshift. We see from Fig.3 that the effects of w_X and Ω_M are of the same order for a large range of redshifts.

In Fig.4 we compare, for Ω_M fixed to 0.3 and for different source redshifts, the compelling effects of α_M and w_X on the distance. When α_M goes away from the usually assumed value ($\alpha_M = 1$), once fixed the redshift, the distance increases; on the contrary, for w_X that

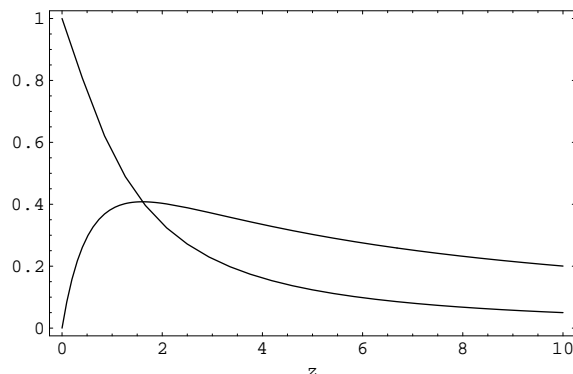


Figure 5: For flat homogeneous universes z_m is determined by the intercept between the angular diameter distance and the always decreasing Hubble distance. The values on the ordinate axis are in units of c/H_0 . It is $\Omega_M = 0.3$, $w_X = -1$.

goes away from the value corresponding to the cosmological constant ($w_X = -1$), the distance decreases. The dependence of the distance on α_M increases very rapidly with z , and, when $z = 5$, the effects of α_M and w_X are of the same order. From Fig.4 we deduce that the dependence on α_M increases when w_X goes to -1 , since values of w_X near zero have the effect to smooth the universe. In fact, when $w_X = -1$, both a fraction α_M of the pressureless matter and of the cosmological constant are uniformly distributed; when $w_X \rightarrow 0$, quintessence behaves like ordinary matter, and so, for the same value of α_M , the pressureless matter homogeneously distributed is $\alpha_M\Omega_M + \Omega_X = 1 - (1 - \alpha_M)\Omega_M$. Intermediate values of w_X interpolate between these two extreme cases.

6 The critical redshift

The critical redshift at which the angular diameter distance of an extragalactic source takes its maximum value has already been studied for the case of a flat Λ CDM universe by Krauss & Schramm (Krauss & Schramm 1993) and for a flat universe with quintessence by Lima & Alcaniz (Lima & Alcaniz 2000). In this section, we will find again their results with a new approach and will extend the analysis to inhomogeneous flat universes.

As can be seen cancelling out the derivative of the right hand of equation (28) with respect to z , the critical redshift z_m for a flat homogeneous universe occurs when

$$D_A(z_d, z_m) = \frac{c}{H(z_m)}, \quad (64)$$

so that the angular diameter distance between an observer at $z = z_d$ and a source at z_m is equal to the Hubble distance for $z = z_m$, as you can see in Fig.5. Equation (64) is an implicit relation that gives the dependence of z_m on z_d, Ω_M and w_X . Throughout this section, we will put $z_d = 0$. In Fig.6 we show z_m for a homogeneous flat universe. For a

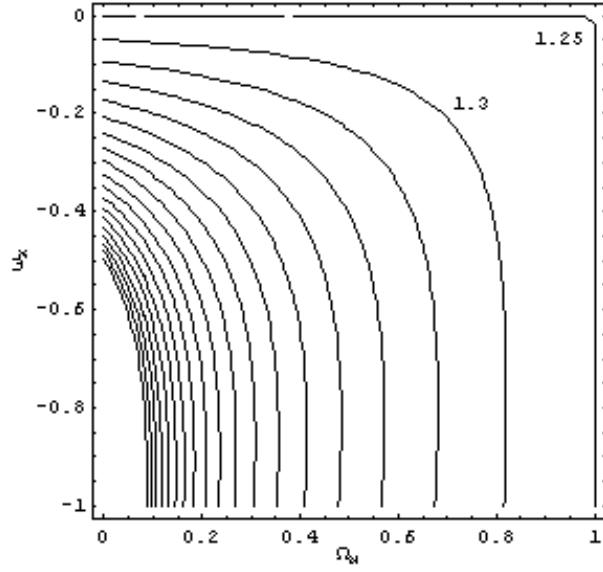


Figure 6: Contours of equal z_m on the $\Omega_M - w_X$ plane for flat homogeneous universes. Each contour is drawn with a step of 0.05.

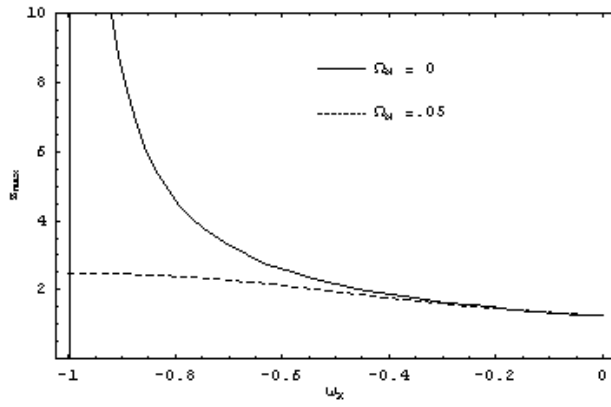


Figure 7: z_m as a function of w_X for two values of Ω_M . For $\Omega_M = 0.05$, z_m nearly halves itself (from 2.47 to 1.25) when w_X goes from -1 to 0 .

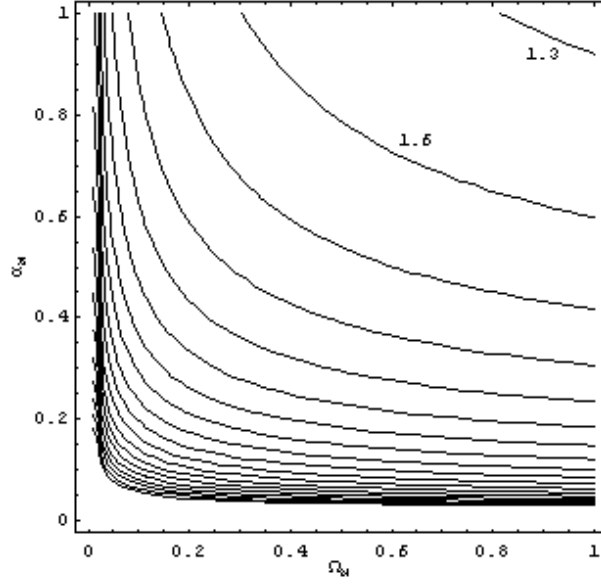


Figure 8: Contours of equal z_m on the $\Omega_M - \alpha_M$ plane for universes with a cosmological constant. Contours are drawn with steps of 0.03.

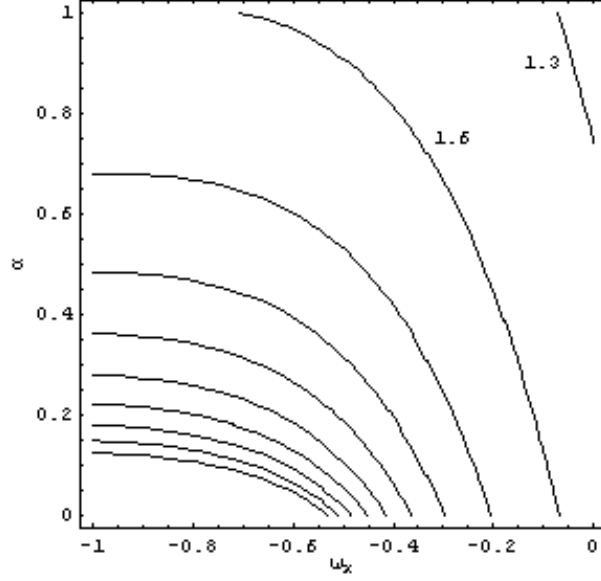


Figure 9: Contours of equal z_m on the $w_X - \alpha_M$ plane for flat universes with $\Omega_M = 0.3$ and $\alpha_X = 1$. Contours are drawn with steps of 0.03.

given value of w_X (Ω_M), z_m decreases with increasing Ω_M (w_X); when $\Omega_M = 0$, z_m diverges for $w_X = -1$, but also a small value of Ω_M is sufficient to eliminate this divergence (see Fig.7). The minimum value of z_m corresponds to the Einstein-de Sitter universe ($\Omega_M = 1$ or $w_X = 0$), when $z_m = 1.25$. As you can see from Fig.6, for values of w_X in the range $(-1, -0.8)$, once fixed Ω_M , z_m is nearly constant and this trend increases with Ω_M ; on the contrary, for small Ω_M ($\lesssim 0.4$) and $w_X \gtrsim -0.4$, z_m is very sensitive to w_X . The small changes of z_m in the region of large Ω_M and w_X are explained with considerations analogous to those already made in the previous section for the values of the distance in the $\Omega_M - w_X$ plane.

Let us go now to analyse the effect of α_M on z_m . By differentiating equation (33) and equation (34) with respect to z , we see that the derivatives are zero only for $z \rightarrow \infty$: i.e., in flat universes with totally inhomogeneous quintessence or in a generic model with cosmological constant, the critical redshift is not finite when $\alpha_M = 0$. So with respect to z_m , a totally clumpy universe, independently of Ω_M and w_X , behaves like a FLRW model completely dominated by the vacuum energy. In fact, the cosmological constant, differently from dark energy with $w_X > -1$, does not give contribution to the Ricci focusing and the same occurs for the pressureless matter with $\alpha_M = 0$. In Fig.8 we show z_m in the $\Omega_M - \alpha_M$ plane for w_X fixed to -1 . The critical redshift decreases with increasing Ω_M and α_M , and takes its minimum for the Einstein-de Sitter universe ($\Omega_M = \alpha_M = 1$), that is when the focusing is maximum. On the other side, z_m is very sensitive to α_M , especially for large values of Ω_M since α_M appears in the DR equation as a multiplicative factor of Ω_M . For $\Omega_M = 0.3$, $z_m = 1.61$ and 3.23 for, respectively, $\alpha_M = 1$ and 0.2 , a variation of 100%. So, combining different cosmological tests to constrain the other cosmological parameters, we can use the redshift-distance relation to guess the smoothness parameter α_M in a quite efficient way.

We conclude this section comparing the influence of α_M and w_X on the critical redshift. Fig.9 displays z_m in the $\alpha_M - w_X$ plane, for Ω_M fixed to 0.3 and with $\alpha_X = 1$. As expected, z_m increases when the focusing decreases, that is for small values of α_M and w_X . We can see that the effects of α_M and Ω_M are of the same order.

7 Conclusions

The question of light propagation in an inhomogeneous universe is an open topic of modern cosmology and the necessity of a positive solution grows up with the increasing means of experimental technology that begins to explore very high redshifts. In particular, the cosmological distances, which provide important probes of the universe, are very sensitive to cosmological inhomogeneities. In fact, in the present quite poor data samples, sources usually appear dimmer and smaller with respect to homogeneous models for the systematic effect of under-densities along the different lines of sight. Here, we have investigated the properties of the angular diameter distance in presence of either clumpy pressureless matter and inhomogeneous dark energy using the 'empty beam approximation'.

The equation for the angular diameter distance with respect to the redshift has been

found in a way that is independent from the focusing equation. The multiple lens-plane theory allows to derive the DR equation for inhomogeneous quintessence, in a way that makes clear the importance of the 'empty beam approximation' in the gravitational lensing.

We have given useful forms for the distance. For non flat universe, we have studied the case of cosmic string network, when the angular diameter distance is expressed in terms of hypergeometric functions, and an accelerated universe with $w_X = -2/3$ (domain walls), when the distance-redshift relation is given in terms of Heun functions. For the very interesting case of flat universes with inhomogeneous quintessence, we have obtained the solution of the DR equation in terms of hypergeometric functions. Then, we have listed the expressions for the distance when it takes the form of elementary functions for two particular cases of flat models: strictly homogeneous (we have considered cosmic string networks again) and totally inhomogeneous universes, i.e. both pressureless matter and dark energy solely in clumps.

As it could be reasonably expected, the angular diameter distance is degenerate, also considering a smooth quintessence, with respect to cosmological parameters: although there is a strong dependence both on the smoothness parameter α_M and the equation of state w_X , the intrinsic systematic errors prevent us to univocally determine the cosmological parameters only from measurements of distance. Nevertheless, combining these data with independent observations like that concerning the cosmic microwave background radiation, we can estimate the smoothness parameter α_M by determining the critical redshift, for example.

The effect of w_X and α_M , especially at high redshifts, are of the same order but very different. An increase in w_X mimics a larger Ω_M , while a more inhomogeneous universe (small α_M) mimics the effect of a cosmological constant. This last statement agrees with what found by C  l  rier (C  l  rier 2000), in a very different context, that is by relaxing the hypothesis of large scale homogeneity for the universe. We have obtained a similar result by adopting the cosmological principle.

Acknowledgments

The authors wish to thank G. Platania, C. Rubano and P. Scudellaro for discussions and valuable comments on the manuscript.

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